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Extended Graetz problem including streamwise conduction and viscous dissipation in microchannel

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Abstract

Extended Graetz problem in microchannel is analyzed by using eigenfunction expansion to solve the energy equation. The hydrodynamically developed flow is assumed to enter the microchannel with uniform temperature or uniform heat flux boundary condition. The effects of velocity and temperature jump boundary condition on the microchannel wall, streamwise conduction and viscous dissipation are all included. From the temperature field obtained, the local Nusselt number distributions are shown as the dimensionless parameters (Peclet number, Knudsen number, Brinkman number) vary. The fully developed Nusselt number for each boundary condition is obtained also in terms of these parameters.

Keywords: Slip flow; Viscous dissipation; Streamwise conduction; Eigenvalue problem; Knudsen number; Peclet number; Brinkman number

1. Introduction

Recently, the electronic system becomes smaller and more complex due to the rapid development of semiconductor technology. The trend of high speed and density of electronic system can lead to serious problem in heat transfer, since large amount of heat per unit area must be dissipated. A great deal of research about micromachines such as microchannel are being carried out to discover the effective cooling technique. However, many experiments have shown that fluid flow and heat transfer characteristics in microtube and microchannel deviate from the well known traditional approaches based on the continuum flow assumption [1,2].

As the size of a channel is reduced, the no-slip boundary conditions need to be modified so that velocity slip and temperature jump may occur on the wall. The slip boundary condition may be used when gases are at low pressure or for flow in extremely small passages. The rarefaction effects of a gas are included by the Knudsen number Kn, the ratio of the mean free path to the characteristic length in the flow field. Karniadakis and Beskok [3] have proposed the range for the Knudsen number in slip flow regime as $0.001 < Kn < 0.1$.

The Graetz problem is a simplified problem of forced convection heat transfer in a circular tube in laminar flow, which is first solved by Graetz [4] analytically assuming fully developed laminar flow and neglecting streamwise heat conduction and viscous dissipation. Sellars et al. [5] extended the Graetz problem including a more effective approximation technique for evaluation of the eigenvalues problem. Lahjomri et al. [6] solved the problem to include streamwise conduction. Barron et al. [7] and Ameel et al. [8] presented an analytical solution with uniform temperature and uniform heat flux boundary conditions, respectively. Tunc and Bayazitoglu [9,10] solved the energy equation with slip velocity and temperature jump boundary conditions in a microtube and a rectangular microchannel analytically, considering viscous dissipation but neglecting the streamwise conduction. Recently, Nield

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Nomenclature

A_n	coefficients
Br	Brinkman number: $Br = \mu u_m^2/k(T_0 - T_w)$, $\mu u_m^2/(q_w H)$
C_1	$1 + 12\frac{2-F}{F}Kn$
C_2	$\frac{2-F_t}{F_t}\frac{2\gamma}{\gamma+1}\frac{Kn}{Pr}$
c_p	specific heat
D_h	hydraulic diameter ($D_h = 4H$)
F	tangential momentum accommodation coefficient
F_t	thermal accommodation coefficient
h	heat transfer coefficient
H	half of channel height
k	thermal conductivity
Kn	Knudsen number ($Kn = \lambda/D_h$)
L	channel length
Nu	Nusselt number ($Nu = hD_h/k$)
P	pressure
Pe	Peclet number ($Pe = Re \cdot Pr = u_m D_h/\alpha$)
Pr	Prandtl number ($Pr = \nu/\alpha$)
q	Heat flux
Re	Reynolds number ($Re = u_m D_h/\nu$)
T	temperature
u	fluid velocity
x, y	Cartesian coordinates
Y_n	eigenfunction
W	channel width

Greek symbols

α	thermal diffusivity
β	eigenvalue
γ	specific heat ratio
λ	molecular mean free path
μ	dynamic viscosity
ν	kinematic viscosity
θ	dimensionless temperature $\theta = (T - T_w)/(T_0 - T_w)$, $k(T - T_0)/(q_w H)$

Subscripts

m	mean values
s	fluid properties at the wall
w	wall values
0	inlet properties
∞	infinite properties

Superscript

*	dimensionless variables

et al. [11–14] investigated thermally developing forced convection in channel or circular duct occupied by porous medium. In the most analysis of the Graetz problems extended, the effects of streamwise conduction and viscous dissipation are not included simultaneously. But, if we use working fluid of high conductivity in heat exchangers and fluid velocity is high, we have to consider both streamwise conduction and viscous dissipation. In this paper, we analyze the extended Graetz problem in a flat channel including effects of rarefaction, streamwise conduction and viscous dissipation altogether. The flow is assumed to be fully developed Poiseuille flow while the temperature profile is just being developed. Two types of heat boundary condition on the wall, isothermal and constant heat flux, are considered. By the results, temperature distributions in the channel are determined with parameters (Knudsen number Kn, Peclet number Pe, and Brinkman number Br) and Nusselt number distributions on the wall are shown for some specific values of parameters. Nusselt numbers for thermally developed state far down stream in the channel are obtained as functions of the parameters.

2. Analysis

2.1. Uniform temperature on the walls

The steady-state hydrodynamically developed flow with constant temperature T_0 enters into the microchannel as illustrated in Fig. 1. The fluid temperature would change from the value T_0 at the entrance to the value T_w on the walls. Assuming laminar incompressible flow, the governing energy equation and boundary condition can be established as

$$\rho c_p u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad (1)$$

$$T = T_0 \quad \text{at } x = 0 \quad (2)$$

$$T - T_w = \mp \frac{2 - F_t}{F_t}\frac{2\gamma}{\gamma + 1}\frac{\lambda}{Pr}\frac{\partial T}{\partial y} \quad \text{at } y = \pm H \quad (3)$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = 0 \quad (4)$$

where u is the fully developed velocity profile in the channel,

$$u(y) = -\frac{H^2}{2\mu}\frac{dP}{dx}\left[1 - \left(\frac{y}{H}\right)^2 + 8\frac{2 - F}{F}Kn\right] \quad (5)$$

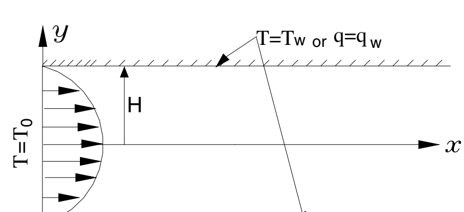

Fig. 1. Definition sketch.

which satisfies the slip boundary condition

$$u = \mp \frac{2-F}{F}\lambda\left(\frac{\partial u}{\partial y}\right) \quad \text{at } y = \pm H \tag{6}$$

We define dimensionless variables

$$\theta = \frac{T - T_{\mathrm{w}}}{T_0 - T_{\mathrm{w}}}, \quad x^* = \frac{x}{Re \cdot Pr \cdot H}, \quad y^* = \frac{y}{H},$$
$$Br = \frac{\mu u_{\mathrm{m}}^2}{k(T_0 - T_{\mathrm{w}})}$$

then dimensionless form of Eq. (1) is

$$\frac{1}{4}u^*\frac{\partial\theta}{\partial x^*} = \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial x^{*2}} + \frac{\partial^2\theta}{\partial y^{*2}} + Br\cdot\left(\frac{\partial u^*}{\partial y^*}\right)^2 \tag{7}$$

where

$$u^* = \frac{u}{u_{\mathrm{m}}} = \frac{3}{2}\frac{(1 - y^{*^2} + 8\frac{2-F}{F}Kn)}{C_1} \tag{8}$$

By the symmetry of the flow field and boundary conditions, we may confine our flow region to $0 \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant 1$. For convenience, we abbreviate the symbol $*$ hereafter. Then, the governing equation and boundary conditions are

$$\frac{1}{4}u\frac{\partial\theta}{\partial x} = \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + Br\cdot\left(\frac{\partial u}{\partial y}\right)^2 \tag{9}$$

$$\theta = 1 \quad \text{at } x = 0 \tag{10}$$

$$\theta = -4C_2\frac{\partial\theta}{\partial y} \quad \text{at } y = 1 \tag{11}$$

$$\frac{\partial\theta}{\partial y} = 0 \quad \text{at } y = 0 \tag{12}$$

As $x \to \infty$, Eq. (9) becomes

$$\frac{\partial^2\theta}{\partial y^2} = -Br\left(\frac{\partial u}{\partial y}\right)^2 \tag{13}$$

since $\frac{\partial\theta}{\partial x} \to 0$.

The fully developed dimensionless temperature profile θ_∞ can be derived by integrating both sides of Eq. (13) with boundary conditions (Eqs. (11) and (12))

$$\theta_\infty(y) = \frac{Br}{C_1^2}\left\{-\frac{3}{4}(y^4 - 1) + 12C_2\right\} \tag{14}$$

Now, we set

$$\theta(x,y) = \theta_1(x,y) + \theta_\infty(y) \tag{15}$$

then $\theta_1 \to 0$ as $x \to \infty$.

Substituting Eq. (15) into Eq. (9), we get

$$\frac{1}{4}u\frac{\partial\theta_1}{\partial x} = \frac{1}{Pe^2}\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_1}{\partial y^2} \tag{16}$$

Note that Eq. (16) is homogeneous and the method of separation of variables may be used. Let $\theta_1(x,y) = \sum_{n=1}^{\infty}A_nX_n(x)Y_n(y)$ then we obtain two ordinary differential equations

$$X_n'(x) + \beta_n^2X_n = 0 \tag{17}$$

$$Y_n''(y) + \beta_n^2\left[\frac{\beta_n^2}{Pe^2} + \frac{1}{4}u\right]Y_n = 0 \tag{18}$$

with boundary conditions

$$Y_n'(0) = 0, \quad Y_n(1) = -4C_2Y_n'(1) \tag{19}$$

where β_n is eigenvalue associated with the eigenfunction $Y_n(y)$. To obtain β_n, $Y_n(y)$ $(n = 1,2,3,\ldots)$ numerically, the shooting method for Eq. (18) is used. The eigenfunctions $Y_n(y)$ $(n = 1,2,3,\ldots)$ are not orthogonal unless $Pe \to \infty$. The temperature distribution $\theta(x,y)$ may now be written as eigenfunction series expansion.

$$\theta(x,y) = \theta_\infty(y) + \sum_{n=1}^{\infty}A_n\exp[-\beta_n^2x]Y_n(y) \tag{20}$$

The unknown coefficients A_n in Eq. (20) are determined from the inlet boundary condition (10).

$$\sum_{n=1}^{\infty}A_nY_n(y) = 1 - \theta_\infty(y) \tag{21}$$

To determine unknown coefficients A_n from Eq. (21), we truncate the infinite series and use the Galerkin method to minimize square of the error from Eq. (21) in $0 \leqslant y \leqslant 1$. From the coefficients A_n calculated, dimensionless temperature distribution in the channel is determined as Eq. (20). The bulk mean temperature (θ_{m}) and Nusselt number at the channel wall (Nu) may be calculated, respectively, as

$$\theta_{\mathrm{m}}(x) = \int_0^1 u(y)\cdot\theta(x,y)\,\mathrm{d}y \tag{22}$$

$$Nu(x) = \frac{h(x)D_{\mathrm{h}}}{k} = -\frac{4}{\theta_{\mathrm{m}}(x)}\frac{\partial\theta}{\partial y}\bigg|_{y=1} \tag{23}$$

where velocity profile $u(y)$ is given in Eq. (8). The average convective heat transfer coefficient and the total heat flux rate from the channel may be expressed as

$$\bar{h} = \frac{1}{L}\int_0^L h(x)\,\mathrm{d}x, \quad q = \bar{h}\cdot W\cdot L\cdot\Delta\theta_{lm} \tag{24}$$

where

$$\Delta\theta_{lm} \equiv \frac{\theta_{\mathrm{m}}(0) - \theta_{\mathrm{m}}(L)}{\ln\frac{\theta_{\mathrm{m}}(0)}{\theta_{\mathrm{m}}(L)}} = \frac{\theta_{\mathrm{m}}(L) - 1}{\ln\theta_{\mathrm{m}}(L)},$$

since $\theta_{\mathrm{m}}(0) = 1$.

2.2. Uniform heat flux on the walls

When the heat flux q_{w} is given on the both of channel walls as illustrated Fig. 1, the following dimensionless variables are redefined as

$$\theta = \frac{k(T - T_0)}{q_{\mathrm{w}}H}, \quad Br = \frac{\mu u_{\mathrm{m}}^2}{q_{\mathrm{w}}H} \tag{25}$$

since the temperature of channel wall is not constant. Substituting Eq. (25) into energy Eq. (1) yields

$$\frac{1}{4}u\frac{\partial\theta}{\partial x}=\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+Br\cdot\left(\frac{\partial u}{\partial y}\right)^2 \tag{26}$$

and boundary conditions are

$$\theta=0\quad \text{at } x=0 \tag{27}$$

$$\frac{\partial\theta}{\partial y}=0\quad \text{at } y=0 \tag{28}$$

$$\frac{\partial\theta}{\partial y}=1\quad \text{at } y=1 \tag{29}$$

Note that governing Eq. (26) and boundary condition (29) are non-homogeneous. Therefore, we introduce a new variable θ_1, such that

$$\theta(x,y)=\theta_1(x,y)+\theta_\infty(x,y) \tag{30}$$

where

$$\theta_\infty(x,y)=4\left[1+\frac{3Br}{C_1^2}\right]x+\frac{y^2}{8}\left[1+\frac{3Br}{C_1^2}\right]\frac{4C_1+2-y^2}{C_1}-\frac{3Br}{4}\frac{y^4}{C_1^2} \tag{31}$$

Substituting Eq. (30) into Eq. (26)–(29), we obtain

$$\frac{1}{4}u\frac{\partial\theta_1}{\partial x}=\frac{1}{Pe^2}\frac{\partial^2\theta_1}{\partial x^2}+\frac{\partial^2\theta_1}{\partial y^2} \tag{32}$$

$$\theta_1(0,y)=-\theta_\infty(0,y)\quad \text{at } x=0 \tag{33}$$

$$\frac{\partial\theta_1}{\partial y}=0\quad \text{at } y=0,1 \tag{34}$$

Since governing Eq. (32) and boundary conditions (34) are now homogeneous, we constitute eigenvalue problem in similar way as the previous section.

$$X'(x)+\beta^2X=0 \tag{35}$$

$$Y''(y)+\beta^2\left(\frac{\beta^2}{Pe^2}+\frac{1}{4}u\right)Y=0 \tag{36}$$

$$Y'(0)=Y'(1)=0 \tag{37}$$

We can calculate eigenvalues and eigenfunctions using the shooting method. Here, it should be mentioned that $\beta=0$ is one of the eigenvalues and corresponding eigenfunction is 1. Finally, we write dimensionless temperature profile as

$$\theta(x,y)=\theta_\infty(x,y)+A_0+\sum_{n=1}^{\infty}A_n\exp(-\beta_n^2x)Y_n(y) \tag{38}$$

The Nusselt number is now easily determined as follows:

$$Nu(x)=\frac{h(x)D_\mathrm{h}}{k}=\frac{4}{\theta_\mathrm{w}-\theta_\mathrm{m}} \tag{39}$$

since $\frac{\partial\theta}{\partial y}=1$ at $y=1$. In Eq. (39), θ_m is a bulk mean temperature as defined in Eq. (22). The wall temperature θ_w in Eq. (39) is given by

$$\theta_\mathrm{w}(x)=\theta(x,1)+4C_2 \tag{40}$$

3. Results and discussion

We first compared our results with those of the classical Graetz problem ($Pe\to\infty$, $Br=Kn=0$) to verify the validity of present calculations. In the calculation, we take $Pe=10^6$ instead of $Pe=\infty$ to see the cases where streamwise conduction term is neglected in energy equation. The results show quite good agreement, which means that eigenvalues and eigenfunctions calculated in the previous section are very accurate. In this work, we found that it is possible to determine as many eigenvalues and eigenfunctions as required. As results, we consider rarefaction effect at the wall by varying Kn and by varying the value of Pe, Br, the effects of the streamwise conduction and viscous dissipation are examined. We carried out the calculation for $Kn=0.04$, 0.08, since slip boundary conditions may be used in $0.001<Kn<0.1$. We assume that working fluid is air, so $Pr=0.7$, $\gamma=1.4$ are used in calculations.

3.1. Uniform temperature on the walls

In Fig. 2, the effects of Knudsen number on heat transfer neglecting streamwise conduction and viscous dissipation are shown. For $Kn=0$, fully developed Nusselt number approximates to 7.54 which is result of the classical Graetz problem. The fully developed Nusselt number decreases as Kn increases. This is due to the fact that the temperature jump on the wall increases and the temperature gradient on the wall decreases as Kn increases. The temperature jump distributions on the wall along the channel wall are shown in Fig. 3 for some Knudsen numbers.

Fig. 4 shows the effect of streamwise conduction on heat transfer neglecting viscous dissipation and slip effect. To show the change with Pe, the abscissa represents x/H in Fig. 4 instead of $x/(Pe\cdot H)$ in other figures. As Pe

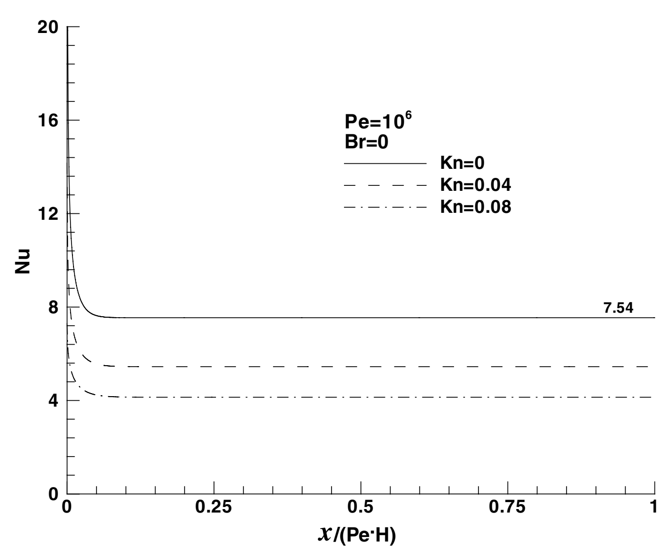

Fig. 2. Nusselt number distributions on the wall for uniform temperature boundary condition. Rarefied effects are considered neglecting viscous dissipation and streamwise conduction.

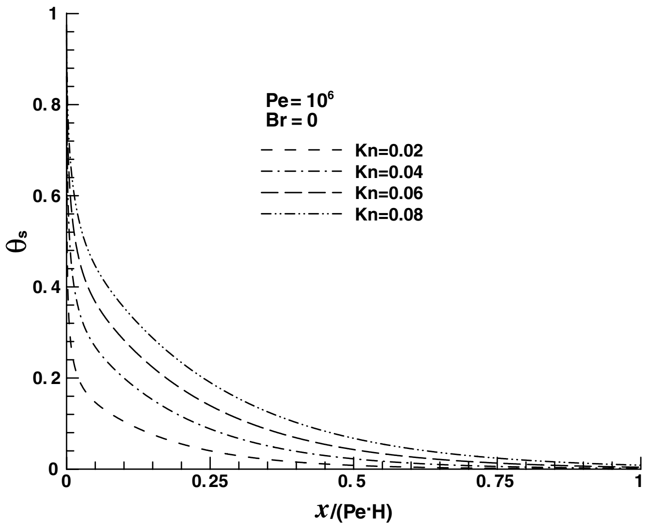

Fig. 3. Temperature jump on the microchannel wall for uniform temperature boundary condition.

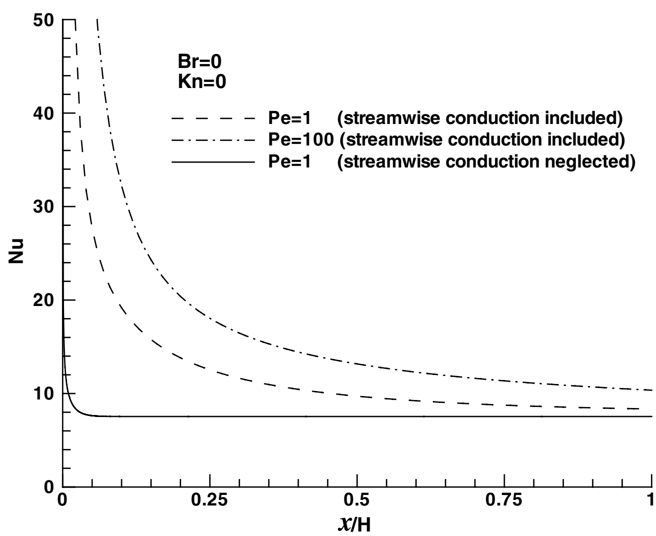

Fig. 4. Nusselt number distributions on the wall for uniform temperature boundary condition. Streamwise conduction effects are considered neglecting viscous dissipation and rarefied effects.

increases, the Nusselt number increases around the inlet of the channel, where convection heat transfer is dominant to the conduction heat transfer. When we take the heat conduction in streamwise x-direction into account, the Nusselt number distribution for $Pe = 1$ is larger than that obtained by neglecting the heat conduction in x-direction as shown in Fig. 4. In other words, the heat transfer to the wall increases if we consider the streamwise heat conduction.

In Fig. 5, we show the effect of viscous dissipation on heat transfer neglecting slip effect and streamwise conduction. The case of $Br = 0$ represents entire neglecting of viscous dissipation in channel. $Br > (<) 0$ means that $T_0 > (<) T_w$ and the fluid is cooled (heated) in the channel. In particular, when $Br < 0$ ($T_0 < T_w$), bulk mean

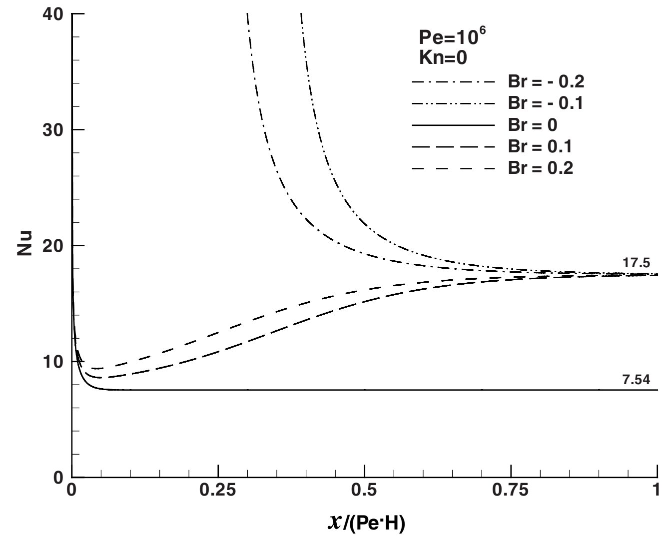

Fig. 5. Nusselt number distribution on the wall for uniform temperature boundary condition. Viscous dissipation effects are considered neglecting streamwise conduction and rarefied effects.

temperature T_m may be equal to the wall temperature T_w at a point $x = x_c$, where Nusselt number is meaningless. This is obvious since T_m increases from T_0 to some temperature above T_w by heat generation due to the viscous dissipation. To verify this explanation, $\theta_m(x)$ is shown in Fig. 6. We notice that the sign of $\theta_m(x)$ changes when $Br < 0$ as mentioned by Nield et al. [14]. Additional typical results for the general cases where three parameters are different from those of the classical Graetz problem are shown in Fig. 7. At $x \to \infty$, the fully developed Nusselt number for $Br \neq 0$ is independent of Br and different from that for $Br = 0$ as shown in Fig. 5. Thermally fully developed Nusselt number may be obtained from the fully developed temperature field as

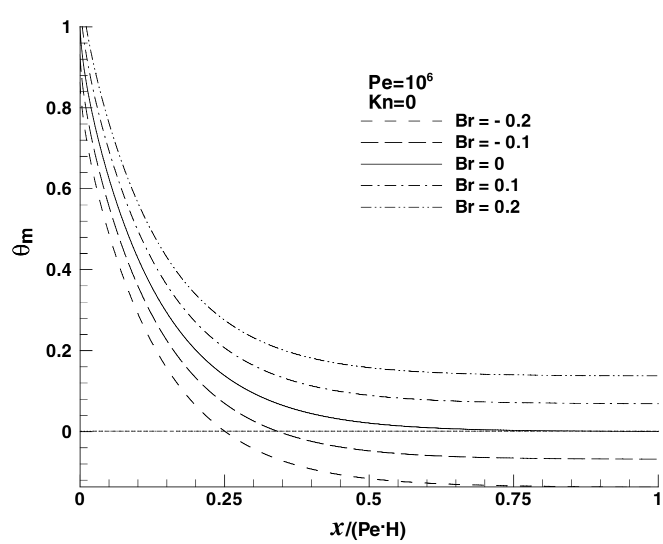

Fig. 6. Bulk mean temperature distributions for uniform temperature boundary condition.

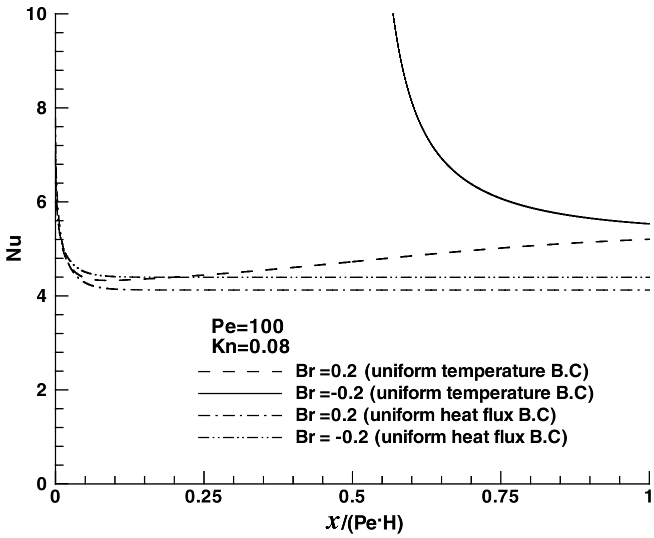

Fig. 7. Nusselt number distributions on the wall for uniform heat flux and temperature boundary condition. Viscous effects, streamwise conduction and rarefied effects are all considered.

$$Nu_\infty = \begin{cases} \dfrac{140C_1}{1+7C_1+140C_1C_2} & \text{for } Br \neq 0 \\ \dfrac{-4Y_1'(1)}{\int_0^1 uY_1(y)\,dy} & \text{for } Br = 0 \end{cases} \tag{41}$$

Note that Nu_∞ for $Br \neq 0$ is analytically determined constant which is independent of Br while Nu_∞ for $Br = 0$ requires numerical calculation.

3.2. Uniform heat flux on the walls

In Fig. 8, we show the effect of Knudsen number on heat transfer with $Pe \to \infty$, $Br = 0$. For $Kn = 0$, the problem reduces to the classical Graetz problem with uniform heat flux boundary condition. The fully developed Nusselt number tends to 140/17 ($\approx$8.24) as $x \to \infty$. As shown in Fig. 8, the fully developed Nusselt number decreases as Kn increases.

Fig. 9 shows the effect of Pe on the heat transfer. Note that the abscissa represents x/H. As illustrated in Fig. 9, the Nusselt number increases as Pe and, for $Pe = 1$, the Nusselt number is larger when we take account of streamwise conduction.

In Fig. 10, the effects of viscous dissipation inside the channel are considered. The case of $Br > 0$ (or $Br < 0$) represents $q_w > 0$ (or $q_w < 0$) and channel is heated (or cooled).

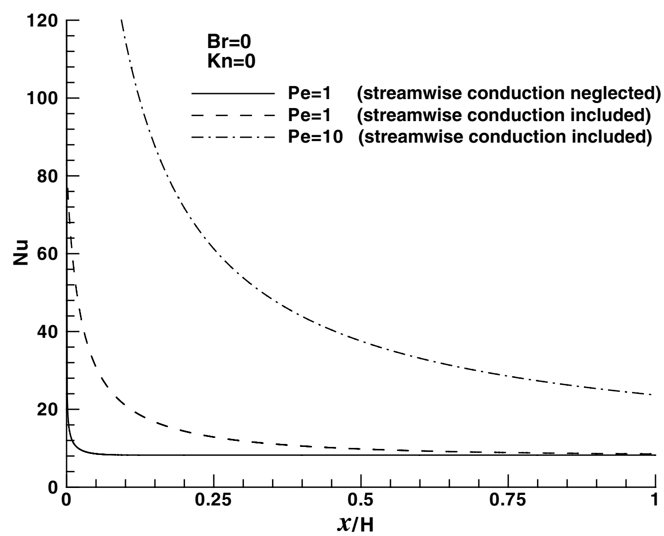

Fig. 9. Nusselt number distributions on the wall for uniform heat flux boundary condition. Streamwise conduction effects are considered neglecting viscous dissipation and rarefied effects.

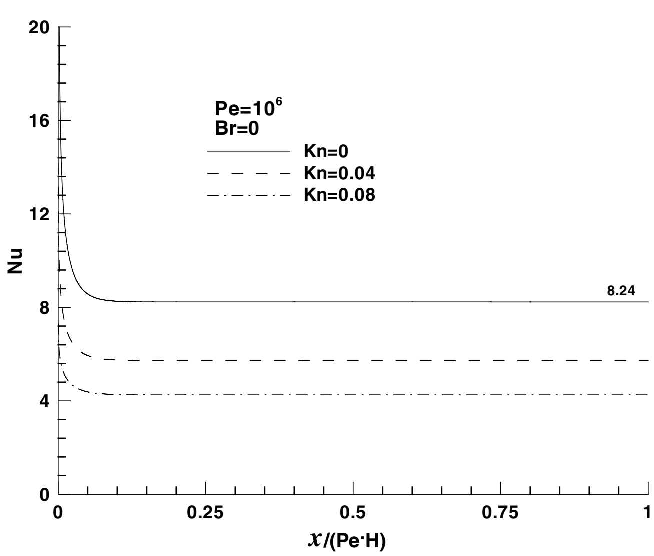

Fig. 8. Nusselt number distributions on the wall for uniform heat flux boundary condition. Rarefied effects are considered neglecting viscous dissipation and streamwise conduction.

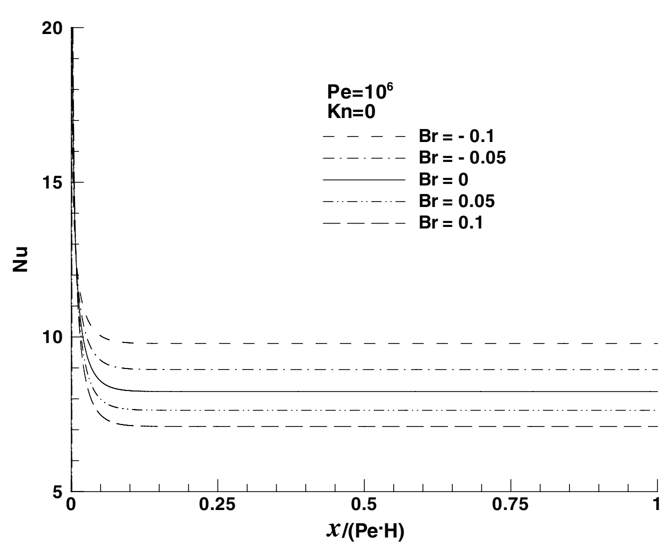

Fig. 10. Nusselt number distributions on the wall for uniform heat flux boundary condition. Viscous effects are considered neglecting streamwise conduction and rarefied effects.

We can see in Fig. 10 that $Nu(x)$ decreases as Br increases. Typical results for the general cases where three parameters are different from those of the classical Graetz problem are also shown in Fig. 7. The fully developed Nusselt number for general case can be derived from Eq. (31), (39), (40) as

$$Nu_\infty = \frac{420C_1^4}{C_1^2(35C_1^2+14C_1+2+420C_1^2C_2)+Br(42C_1^2+33C_1+6)} \tag{42}$$

Unless Br is large with negative sign, $Nu > 0$ always. Temperature jump between the channel wall and adjacent fluid may be easily written as

$$\theta_w(x) - \theta(x,1) = 4C_2 \tag{43}$$

since $\frac{\partial\theta}{\partial y}(x,1) = 1$. This temperature jump is a constant which is proportional to Kn but independent of Pe and Br.

4. Conclusions

We have investigated the effects of rarefaction, streamwise conduction and viscous dissipation on Graetz problem in a flat microchannel with the uniform temperature and the uniform heat flux boundary conditions, respectively, on the walls. The Nusselt number decreases as Knudsen number or Brinkman number increases and as Peclet number decreases. When the streamwise conduction is included, the Nusselt number is larger compared with that of Graetz solution where streamwise conduction is neglected. We have also found the fully developed Nusselt number for the extended Graetz problems in terms of the parameters.

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